DETERMINANTS

- $A \longrightarrow$ Matrices.
  $|A| \longrightarrow$ determinants of A.
- Only square Matrix has determinates.
- We have a certain Value in determinate not in Matrix.
- determinant upto order 3 is comforable for Calculation.
- order of determinant.
  1. 1×1
  2. 2×2.
  3. 3×3.

Eg
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \Delta = ad - bc.$$

Eg
$$\begin{vmatrix} a & b & c \\ x & y & z \\ \alpha & \beta & \gamma \end{vmatrix}$$

$$\Delta = a\begin{vmatrix} y & z \\ \beta & \gamma \end{vmatrix} - b\begin{vmatrix} x & z \\ \alpha & \gamma \end{vmatrix} + c\begin{vmatrix} x & y \\ \alpha & \beta \end{vmatrix}$$

$$\Delta = a(y\gamma - z\beta) - b(x\gamma - \alpha z) + c(x\beta - \alpha y).$$

Q Find the Δ

① $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$

② $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

NOTE $(-1)^{i+j}$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Sol
$$= \begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$$
$$= (x-1)(x^2+x+1) - x^3$$
$$= -1 \text{ Ans}$$

②
$$\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$
$$= 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$
$$= 3(1+6) + 4(1+4) + 5(3-2)$$
$$= 21 + 20 + 5 = 46 \text{ Ans}$$

Q Evaluate x if $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$.

Sol

$$2-20 = 2x^2-24.$$

$$-18 = 2x^2-24 \rightsquigarrow 2x^2 = 24-18 = 6 \therefore x^2 = 3.$$

$$x = \pm\sqrt{3} \text{ Ans}$$

Q Find Δ

$$\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$$

Sol $0 - \sin\alpha[0 - \sin\beta\cos\alpha] - \cos\alpha[\sin\alpha\sin\beta - 0] = 0.$

Properties of Determinants :-

① $R \leftrightarrow C \rightarrow \Delta$ will remain same.

② If any two Rows or Columbs of a determinant are Interchanged the the Δ becomes $-$ve.

Ex $R_1 \leftrightarrow R_2$ $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

③ If any two Rows or Columbs are same or proporsonal then $\Delta = 0$.

Ex $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 5 & 6 & 7 \end{vmatrix} : \Delta = 0.$ Ex $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 10 & 12 & 14 \end{vmatrix} : \Delta = 0$ $R_3 \leftrightarrow 2R_2$.

④ We can take common from any row or Columb.

$$\begin{vmatrix} a & b & c \\ Kx & Ky & Kz \\ \alpha & \beta & \gamma \end{vmatrix} = K\begin{vmatrix} a & b & c \\ x & y & z \\ \alpha & \beta & \gamma \end{vmatrix}$$

⑤ If the entire row or Columb is zero then $\Delta = 0$.

$$\begin{vmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} : \Delta = 0.$$

⑥ Sum of Row or Columb.

$$\begin{vmatrix} a & b & c \\ x & y & z \\ d+e & f+g & h+i \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ d & f & h \end{vmatrix} + \begin{vmatrix} a & b & c \\ x & y & z \\ e & g & i \end{vmatrix}$$

Q Without expanding show that.

① $\begin{vmatrix} 102 & 1 & 17 \\ 18 & 3 & 3 \\ 36 & 4 & 6 \end{vmatrix} = 0$ ② $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Sol ①
$= \begin{vmatrix} 102 & 1 & 17 \\ 18 & 3 & 3 \\ 36 & 4 & 6 \end{vmatrix} = 6 \begin{vmatrix} 17 & 1 & 17 \\ 3 & 3 & 3 \\ 6 & 4 & 6 \end{vmatrix}$

Since C_1 and C_3 are same.
$\therefore 6 \cdot \Delta = 6 \cdot 0 = 0$
Proved.

② $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} \Rightarrow C_2 \to C_2 + C_1 \begin{vmatrix} x & x+a & x+a \\ y & y+b & y+b \\ z & z+c & z+c \end{vmatrix} = 0$ as $C_2 = C_3$

Q Using property prove $\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

Sol $R_2 \to R_2 - R_3$
$R_3 \to R_3 - R_1$

$\begin{vmatrix} 1 & a & bc \\ 0 & (b-c) & a(c-b) \\ 0 & (c-a) & b(a-c) \end{vmatrix} = (b-c)(c-a) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -a \\ 0 & 1 & -b \end{vmatrix}$

$= (b-c)(c-a)\left[1(-b+a)\right]$

$= (b-c)(c-a)(a-b)$ proved

Q Prove $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

Sol $C_1 \to C_1 + C_2 + C_3$

$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix} = (x+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$(x+a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = (x+a+b+c)(x^2 - 0)$

$\Rightarrow x^2(x+a+b+c)$

2014 (CBSE)

Q Prove $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ac + ab$ or $abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

Sol $R_1 \to \frac{1}{a}R_1$, $R_2 \to \frac{1}{b}R_2$, $R_3 = \frac{1}{c}R_3$

$$= abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$R_1 \to R_1+R_2+R_3$

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ 1/b & \frac{1}{b}+1 & 1/b \\ 1/c & 1/c & 1+\frac{1}{c} \end{vmatrix}$$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1)$. Proved $\rightsquigarrow \cancel{abc}\left(\frac{abc+bc+ac+ab}{\cancel{abc}}\right)$ Proved

Q 2015, 2011, 2014 Foreign

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Sol
$$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$C_3 \to C_3 - (C_1+C_2)$

$$= abc\begin{vmatrix} a & c & 0 \\ a+b & b & -2b \\ b & b+c & -2b \end{vmatrix} = abc(-2b)\begin{vmatrix} a & c & 0 \\ a+b & b & 1 \\ b & b+c & 1 \end{vmatrix}$$

$= -2ab^2c$.

$C_1 \to C_1 - bC_3$, $C_2 \to C_2 - bC_3$

$$= -2ab^2c\begin{vmatrix} a & c & 0 \\ a & 0 & 1 \\ 0 & c & 1 \end{vmatrix}$$

$$\Rightarrow -2a^2b^2c\begin{vmatrix} 1 & c & 0 \\ 1 & 0 & 1 \\ 0 & c & 1 \end{vmatrix} = -2a^2b^2c^2\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(-1) - 1(1)\ 0$$

$$= -1 - 1 = (-2)$$

$$\therefore\ 4a^2b^2c^2$$

2017

Q If $\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$ Use the property of determinant, find the Value of $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$.

Sol $\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$

Sol $= xyz\begin{vmatrix} a/x & \frac{b}{y}-1 & \frac{c}{z}-1 \\ \frac{a}{x}-1 & b/y & \frac{c}{z}-1 \\ \frac{a}{x}-1 & \frac{b}{y}-1 & \frac{c}{z} \end{vmatrix} = 0$

then $C_1 \to C_1 + C_2 + C_3$.

$$= xyz\begin{vmatrix} \frac{a}{x}+\frac{b}{y}+\frac{c}{z}-2 & \frac{b}{y}-1 & \frac{c}{z}-1 \\ \frac{a}{x}+\frac{b}{y}+\frac{c}{z}-2 & \frac{b}{y} & \frac{c}{z}-1 \\ \frac{a}{x}+\frac{b}{y}+\frac{c}{z}-2 & \frac{b}{y}-1 & \frac{c}{z} \end{vmatrix} = 0$$

$$xyz\left[\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-2\right]\begin{vmatrix} 1 & \frac{b}{y}-1 & \frac{c}{z}-1 \\ 1 & \frac{b}{y} & \frac{c}{z}-1 \\ 1 & \frac{b}{y}-1 & \frac{c}{z} \end{vmatrix} = 0$$

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-2 = 0.$$

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = 2.$$

Q 2014 Write the Δ of $\begin{vmatrix} p & p+1 \\ p-1 & p \end{vmatrix}$

Sol $\Delta = p^2 - (p+1)(p-1)$

$\Delta = \cancel{p^2} - \cancel{p^2} + 1 = 1.$

Q 2011 Evaluate $\begin{vmatrix} \cos 15 & \sin 15 \\ \sin 75 & \cos 75 \end{vmatrix}$

Sol $\cos 15 \cos 75 - \sin 15 \sin 75.$

$\cos(15+75)$

$\cos 90 = 0$

Q Evaluate $\begin{vmatrix} \log_3 256 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix}$

$\log m^n = n \log m$

$\log_a b = \dfrac{\log b}{\log a}$

Sol
$\log_3 256 \log_4 9 - \log_3 8 \log_4 3$

$8\log_3 2 \cdot 2\log_4 3 - 3\log_3 2 \log_4 3$

$\log_3 2 \cdot \log_4 3 [16-3] = 13 \log_3 2 \cdot \log_4 3 \Rightarrow 13 \dfrac{\cancel{\log 2}}{\cancel{\log 3}} \times \dfrac{\cancel{\log 3}}{2\cancel{\log 2}} = \dfrac{13}{2}$

2013 c

Q If $\begin{vmatrix} 2x & x+3 \\ 2(x+1) & x+1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix}$ Find 'x'.

Sol
$2x(x+1) - 2(x+1)(x+3) = 3-15$

$x = 1$

Q 2015 Find Δ of $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$

Sol
$R_1 \to R_1+R_2 = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix} = (x+y+z)(-3)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

4 2018

Q Using property prove that $\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx)$

Sol
$xyz\begin{vmatrix} \frac{1}{x} & \frac{1}{x} & \frac{1}{x}+3 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix} = R_1 \to R_1+R_2+R_3$

$= xyz\begin{vmatrix} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 \\ \frac{1}{y}+3 & 1/y & 1/y \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$

$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y}+3 & 1/y & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & 1/z \end{vmatrix}$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$(yz + xz + xy + 3xyz)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{y}+3 & -3 & -3 \\ \frac{1}{z} & 3 & 0 \end{vmatrix} \Rightarrow (xy+yz+xz+3xyz)\big(0-(-9)\big) - 0 - 0$$

$$= 9(xy+yz+xz+3xyz)$$

Q 2013C. Using properties of Δ prove. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$.

NOTE

$(x-y)^3 = x^3 - y^3 - 3x^2y + 3xy^2$

$(x-y)^3 + 3x^2y - 3xy^2 = x^3 - y^3$

$(x-y)^3 + 3xy(x-y) = x^3 - y^3$

$(x-y)(x^2+y^2-2xy+3xy)$

$x^3 - y^3 = (x-y)[x^2+y^2+xy]$

Sol

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^3 \\ 0 & (b-a) & (b-a)(b^2+a^2+ab) \\ 0 & (c-a) & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & b^2+a^2+ab \\ 0 & 1 & c^2+a^2+ac \end{vmatrix}$$

$$= (a-b)(a-c)\left[\cancel{a^2}+c^2+ac-b^2-\cancel{a^2}-ab\right]$$

$$= (a-b)(a-c)\left[c^2-b^2+a(c-b)\right]$$

$$= (a-b)(a-c)\left[(c-b)(c+b)+a(c-b)\right]$$

$$= (a-b)(a-c)(c-b)[a+b+c]$$

$$= (a-b)(c-a)(b-c)[a+b+c]$$

Q All India 2017

Prove that $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$.

Sol

$R_1 \to R_1 - R_2$

$R_2 \to R_2 - R_3$

$$\begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} (a-1)(a+1) & (a-1) & 0 \\ 2(a-1) & (a-1) & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$(a-1)^2\begin{vmatrix}(a+1) & 1 & 0\\ 2 & 1 & 0\\ 3 & 3 & 1\end{vmatrix} = (a-1)^2\left[(a+1)[1-0]-1[2-0]+0\right]$$
$$= (a-1)^2[a+1-2] = (a-1)^2[a-1]$$
$$= (a-1)^3$$

Delhi 2015 C.

Q Using properties prove.

$$\begin{vmatrix}(a+1)(a+2) & (a+2) & 1\\ (a+2)(a+3) & (a+3) & 1\\ (a+3)(a+4) & (a+4) & 1\end{vmatrix} = -2.$$

Sol $R_3 \to R_3 - R_2$.
$R_2 \to R_2 - R_1$.

$$= \begin{vmatrix}(a+1)(a+2) & a+2 & 1\\ (a+2)(a+3)-(a+1)(a+2) & (a+3)-(a+2) & 0\\ (a+3)(a+4)-(a+2)(a+3) & (a+4)-(a+3) & 0\end{vmatrix}$$

$$= \begin{vmatrix}(a+1)(a+2) & (a+2) & 1\\ (a+2)[a+3-a-1] & 1 & 0\\ (a+3)[a+4-a-2] & 1 & 0\end{vmatrix}$$

$$= \begin{vmatrix}(a+1)(a+2) & a+2 & 1\\ 2(a+2) & 1 & 0\\ 2(a+3) & 1 & 0\end{vmatrix} = 1\left[2(a+2)-2(a+3)\right]$$
$$= 2[a+2-a-3] = -2 \quad \text{proved}$$

2015 C.

Q Solve x

$$\begin{vmatrix}a+x & a-x & a-x\\ a-x & a+x & a-x\\ a-x & a-x & a+x\end{vmatrix} = 0.$$

Sol $C_1 \to C_1 + C_2 + C_3$: $C_2 \to C_2 - C_3$.

$$\begin{vmatrix}3a-x & 0 & a-x\\ 3a-x & 2x & a-x\\ 3a-x & -2x & a+x\end{vmatrix} = 0 \;\therefore\; 2x(3a-x)\begin{vmatrix}1 & 0 & a-x\\ 1 & 1 & a-x\\ 1 & -1 & a+x\end{vmatrix} = 0$$

$$2x(3a-x) = 0 \longrightarrow x = 0$$
$$\longrightarrow x = 3a$$

Delhi - 2013

Q Prove $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$.

$x^3-y^3 = (x-y)(x^2+y^2+xy)$

Sol $C_1 \to C_1+C_2+C_3$.

$$\begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} \Rightarrow (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

$C_2 \to C_2-C_1$; $C_3 \to C_3-C_1$

$$(1+x+x^2)\begin{vmatrix} 1 & x-1 & x^2-1 \\ 1 & 0 & x-1 \\ 1 & x^2-1 & 0 \end{vmatrix} = (x^2+x+1)\begin{vmatrix} 1 & x-1 & (x-1)(x+1) \\ 1 & 0 & (x-1) \\ 1 & (x-1)(x+1) & 0 \end{vmatrix} = (x-1)^2(1+x+x^2)\begin{vmatrix} 1 & 1 & x+1 \\ 1 & 0 & 1 \\ 1 & x+1 & 0 \end{vmatrix}$$

$$(x-1)^2(1+x+x^2)\begin{vmatrix} 1 & 1 & x-1 \\ 0 & -1 & -x \\ 0 & x & -x-1 \end{vmatrix} = (x-1)^2(x^2+1+x)\left[(x+1)+x^2\right]$$

$C_2 \to C_2-C_1$
$C_3 \to C_3-C_1$

$$= \left[(x-1)(x^2+x+1)\right]^2$$

$$= \left[x^3-(1)^2\right]^2 \quad \text{proved}$$

All India 2011 C.

Q Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$.

Sol $abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$; $R_2 \to R_2-R_1$, $R_3 \to R_3-R_1$ $= abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$

$$= abc \times \frac{1}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 1\left[(a^2+1) - b^2(-1) + c^2(1)\right] = a^2+1+b^2+c^2$$
$$= a^2+b^2+c^2+1 \quad \text{Proved}$$

2017C

⑥ Q If $a+b+c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ prove $a=b=c$.

Sol $R_1 \to R_1 + R_2 + R_3 \Rightarrow (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$

$= (bc-a^2) - 1(b^2-ac) + 1(ab-c^2)$

$= -a^2-b^2-c^2+ac+ab+bc$

$= (a+b+c)\left(-\frac{1}{2}\right)(2a^2+2b^2+2c^2-2ac-2ab-2bc)$

$= -\frac{1}{2}(a+b+c)\left[a^2+a^2+b^2+b^2+c^2+c^2-2ac-2ab-2bc\right] = 0$

$-\frac{1}{2}(a^2+b^2+c^2)\left[(a^2+b^2-2ab)+(a^2+c^2-2ac)+(b^2+c^2-2bc)\right]=0$

$= -\frac{1}{2}\underbrace{(a^2+b^2+c^2)}_{\neq 0}\left[\underbrace{(a-b)^2}_{0}+\underbrace{(a-c)^2}_{0}+\underbrace{(b-c)^2}_{0}\right]=0$

$\therefore$ $a=b=c$. Proved.

⑥ 2016.

Q Prove $\begin{vmatrix} (x+y)^2 & xz & zy \\ zx & (z+y)^2 & xy \\ zy & xy & (z+x)^2 \end{vmatrix} = 2xyz(x+y+z)^3$.

Sol $R_1 \to zR_1$, $R_2 \to xR_2$, $R_3 \to yR_3$

$\begin{vmatrix} z(x+y)^2 & xz^2 & z^2y \\ x^2z & x(z+y)^2 & x^2y \\ zy^2 & xy^2 & y(z+x)^2 \end{vmatrix}$

$= zxy\begin{vmatrix} (x+y)^2 & z^2 & z^2 \\ x^2 & (z+y)^2 & x^2 \\ y^2 & y^2 & (z+x)^2 \end{vmatrix}$

$C_2 \to C_2 - C_1$ $\quad = xyz \begin{vmatrix} (x+y)^2 & z^2-(x+y)^2 & z^2-(x+y)^2 \\ x^2 & (z+y)^2 - x^2 & 0 \\ y^2 & 0 & (z+x)^2 - y^2 \end{vmatrix}$

$C_3 \to C_3 - C_1$

$$= xyz \begin{vmatrix} (x+y)^2 & (z-x-y)(z+x+y) & (z-x-y)(z+x+y) \\ x^2 & (z+y-x)(z+y+x) & 0 \\ y^2 & 0 & (z+x-y)(z+x+y) \end{vmatrix}$$

$$= xyz(x+y+z)^2 \begin{vmatrix} x^2+y^2+2xy & (z-x-y) & (z-x-y) \\ x^2 & (z+y-x) & 0 \\ y^2 & 0 & (z+x-y) \end{vmatrix}$$

$R_1 \to R_1 - (R_2+R_3)$

$$= 2xyz(x+y+z)^2 \begin{vmatrix} xy & -y & -x \\ x^2 & z+y-x & 0 \\ y^2 & 0 & z+x-y \end{vmatrix}$$

$\to C_2 \to C_2 + \frac{C_1}{x}$ $\qquad \to C_3 \to C_3 + \frac{C_1}{y}$

$-y + \frac{1}{\cancel{x}} \cdot \cancel{x}y = 0$ $\qquad \to -x + \frac{1}{\cancel{y}} \cdot x\cancel{y} = 0$

$z+y-x + \frac{1}{\cancel{x}} \cdot x^{\cancel{2}} = z+y$ $\qquad 0 + \frac{1}{y} x^2 = x^2/y$

$0 + \frac{1}{x} \cdot y^2 = y^2/x$ $\qquad z+x-y + \frac{1}{\cancel{y}}(y^{\cancel{2}}) = z+x$

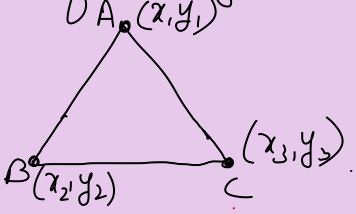
$$= 2xyz(x+y+z)^2 \begin{vmatrix} xy & 0 & 0 \\ x^2 & z+y & x^2/y \\ y^2 & y^2/x & x+z \end{vmatrix}$$

$$= 2xyz(x+y+z)^2 \left[xy \left[(z+y)(x+z) - \left(\frac{x^{\cancel{2}}}{\cancel{y}}\right)\left(\frac{y^{\cancel{2}}}{\cancel{x}}\right) \right] \right]$$

$$= 2xyz(x+y+z)^2 \left[xy \left[\underbrace{xz + \cancel{xy} + z^2 + yz - \cancel{xy}}_{(z^2 + z(x+y))} \right] \right]$$

Use of Determinants in Co-ordinate geometry :-

① Area of triangle.

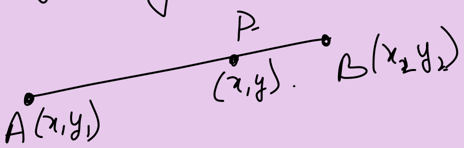


$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

② $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$ ⇝ Then the three co-ordines are Co-linear.

③ Eq of a line.

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two given points. then the Eq of line joining A and B. with any point on the line $P(x,y)$.



$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ x and y Variable.

Q Find the area of Δ whose Vertices are $(3,8)$ $(-4,2)$ and $(5,1)$.

Sol $$\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 8\begin{vmatrix} -4 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} -4 & 2 \\ 5 & 1 \end{vmatrix} = \frac{61}{2} \text{ Sq Units.}$$

Q If the area of a Δ $(-3,0)$ $(3,0)$ & $(0,K)$ Vertices is 9 sq Units. Find the Value of K.

Sol $$\text{Ar}\Delta = \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & K & 1 \end{vmatrix} = \frac{1}{2}\left[-3(0-K) - 0 + 1(3K-0)\right] = 9$$

$$= 3K + 3K = 18$$
$$6K = 18 \quad \boxed{K = 3} \quad \text{Ans}$$

K may be +3 or −3. why?? because Area = 9 is always taken +ve.

Q If the points $(2,-3)$ $(\lambda,-1)$ and $(0,4)$ are Colinear, then find the Value of λ.

Sol
$$\begin{vmatrix} 2 & -3 & 1 \\ \lambda & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix} = 0 \quad \text{Co-linear.} \quad 2(-1-4)+3(\lambda)+1(4\lambda)=0$$
$$-10+7\lambda=0 \qquad \lambda=\frac{10}{7} \text{ Ans}$$

Q Find the Eq of line joining (2,3) and (−1,2).

Sol
$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{vmatrix} = x(3-2)-y(2+1)+1(4+3)=0$$
$$x-3y+7=0 \text{ Ans}$$

Q Find the equation of a line joining P(11,7) and Q(5,5). Also find the Value of K if R(−1,K) is the point such that area of △ PQR is 9 sq m.

Sol
$$\begin{vmatrix} x & y & 1 \\ 11 & 7 & 1 \\ 5 & 5 & 1 \end{vmatrix} = 0 = x(7-5)-y(11-5)+1(55-35)=0$$
$$2x-6y+20=0$$
$$x-3y+10=0 \text{ Ans}$$

(2)
$$\begin{vmatrix} 11 & 7 & 1 \\ 5 & 5 & 1 \\ -1 & K & 1 \end{vmatrix} = \pm 9 \quad = \frac{1}{2}\left[11(5-K)-7(5+1)+1(5K+5)\right]$$
$$55-11K-42+5K+5=\pm 18$$
$$-6K+18=\pm 18$$

$-6K+\cancel{18}=+\cancel{18}$ $\quad \boxed{K=0}$ $\qquad -6K+18=-18$ $\quad \boxed{K=6}$ Ans

Q Find the Value of K, if the points (K+1, 1) (2K+1, 3) and (2K+2, 2K) are Colinear.

Sol
$$\begin{vmatrix} K+1 & 1 & 1 \\ 2K+1 & 3 & 1 \\ 2K+2 & 2K & 1 \end{vmatrix} = 0 \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \qquad \begin{vmatrix} K+1 & 1 & 1 \\ K & 2 & 0 \\ K+1 & 2K-1 & 0 \end{vmatrix}$$

$$= \cancel{(K+1)[0]} - \cancel{1[0]} + 1\left(K(2K-1)-2(K+1)\right) = 0$$
$$2K^2-K-2K-2=0$$
$$2K^2-3K-2=0$$
$$2K^2-4K+K-2=0$$
$$2K(K-2)+1(K-2)=0 \to (2K+1)(K-2)=0$$
$$K=\frac{-1}{2},\ 2 \text{ Ans}$$

MINORS and Co-factors:-

↳ expansion of determinants.

Minors of an element a_{ij} of a determinant is the determinant obtained by deleting ith row and jth Column. M_{ij}

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 2 & -1 & 3 \end{vmatrix}$$

Ex $a_{11} = 2$

$$M_{11} = \begin{vmatrix} 6 & 7 \\ -1 & 3 \end{vmatrix} = 18 + 7 = 25.$$

$a_{23} = 7$

$$M_{23} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = -2-6 = -8.$$

$a_{32} = -1.$

$$M_{32} = \begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix} = 14-20 = -6.$$

NOTE $\begin{vmatrix} -3 & 4 \\ 6 & 5 \end{vmatrix}_{2\times 2}$ $\Delta = -15-24 = -39 = M.$

Co-factor - If M_{ij} of an element a_{ij}, then the Co-factor of a_{ij} is denoted by C_{ij} or A_{ij}

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Ex $\begin{vmatrix} 2 & 0 & 3 \\ 1 & -3 & 4 \\ 7 & 6 & 5 \end{vmatrix}$

- $a_{11} = 2$: $M_{11} = \begin{vmatrix} -3 & 4 \\ 6 & 5 \end{vmatrix} = -15-24 = -39.$: $C_{11} = (-1)^2(-39) = -39.$
- $a_{12} = 0$: $M_{12} = \begin{vmatrix} 1 & 4 \\ 7 & 5 \end{vmatrix} = 5-28 = -23$: $C_{12} = (-1)^3(-23) = 23.$
- $a_{13} = 3$: $M_{13} = \begin{vmatrix} 1 & -3 \\ 7 & 6 \end{vmatrix} = 6+21 = 27$: $C_{13} = (-1)^4(27) = 27.$

Minor ⇓ Co-factor ⇓

$$\begin{vmatrix} -39 & -23 & 27 \\ -18 & -11 & 12 \\ 9 & 5 & -6 \end{vmatrix} \Longrightarrow \begin{vmatrix} -39 & +23 & 27 \\ +18 & -11 & -12 \\ 9 & -5 & -6 \end{vmatrix}$$

Q Using Co-factor of Third Row evaluate

$$\Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

Sol

$$\Delta = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$= (3+\sqrt{115})\left[(-1)^{3+1}\begin{vmatrix} \sqrt{5} & \sqrt{5} \\ 5 & \sqrt{10} \end{vmatrix}\right] + \sqrt{15}\left[(-1)^{3+2}\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & \sqrt{10} \end{vmatrix}\right]$$

$$+ 5\left[(-1)^{3+3}\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 \end{vmatrix}\right]$$

$$\rightarrow = (3+\sqrt{115})\left[\sqrt{50}-5\sqrt{5}\right] - \sqrt{15}\left[\cancel{\sqrt{230}}+\sqrt{30} - \sqrt{75} - \cancel{\sqrt{230}}\right] + 5\left[5\sqrt{23} + 5\sqrt{3} - \sqrt{75} - \sqrt{230}\right]$$

$$= 3\sqrt{50} - 15\sqrt{5} + \sqrt{5750} - 5\sqrt{575} - \sqrt{450} + \sqrt{1125} + 25\sqrt{23} + 25\sqrt{3} - 5\sqrt{75} - 5\sqrt{230}$$

$$= \cancel{15\sqrt{2}} - \cancel{15\sqrt{5}} + \cancel{5\sqrt{230}} - \cancel{25\sqrt{23}} - \cancel{15\sqrt{2}} + \cancel{15\sqrt{5}} + \cancel{25\sqrt{23}} + \cancel{25\sqrt{3}} - \cancel{25\sqrt{3}} - \cancel{5\sqrt{230}}$$

$$= 0$$

Q Find minor and Co-factors of the elements of the determinant

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

and Verify $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$

Sol

$$A = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

Major $= M = \begin{vmatrix} -20 & -46 & 30 \\ -4 & -19 & 13 \\ -12 & -22 & 18 \end{vmatrix}$

$a_{11} = 2$	$C_{31} = -12$
$a_{12} = -3$	$C_{32} = 22$
$a_{13} = 5$	$C_{33} = 18$

Co-factor $C = \begin{vmatrix} -20 & +46 & 30 \\ +4 & -19 & -13 \\ -12 & +22 & 18 \end{vmatrix}$

$$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33}$$

$$2\times(-12) + (-3)(22) + 5\times 18$$

$$-24 - 66 + 90 = 0 \quad \text{Ans}$$

$$\text{Adj } A = \begin{vmatrix} -20 & 4 & -12 \\ 46 & -19 & 22 \\ 30 & -13 & 18 \end{vmatrix}$$

Adj A − Transpose of Co-factor.

ADJOINT AND INVERSE of a MATRIX

- **Singular and non Singular Matrix:–**

  Singular matrix. $|A| = 0$.

  non Singular Matrix $|A| \neq 0$.

- **Adjoint of a Matrix :–**

  – Transpose of the matrix formed by Co-factors.

  – adj (A) representation.

  – For 2x2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \Rightarrow Adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Q Find the adjoint of the matrix.

(1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (2) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Sol $Adj(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ Ans

(2) $A \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix} \Rightarrow$ Co-factor $= \begin{bmatrix} 3 & -12 & 6 \\ +1 & 5 & +2 \\ -11 & -1 & 5 \end{bmatrix}$

$Adj(A) = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$ Ans

2015

Q Find the adjoint of the matrix $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ and hence show that $A\,(adj(A)) = |A|\, I$.

Sol $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

LHS

Co-factor – $\begin{bmatrix} -3 & -6 & -6 \\ +6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$

$Adj(A) = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$

RHS

$|A| = -1(1-4) + 2(2+4) - 2(-4-2)$

$= 3 + 12 + 12 = 27$.

$\rightarrow 27\, I$.

$$A \cdot Adj(A) = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+12+12 & -6-6+12 & -6+12-6 \\ -6-6+12 & 12+3+12 & 12-6-6 \\ -6+12-6 & 12-6-6 & 12+12+3 \end{bmatrix} = \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$= 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 27I \longrightarrow LHS.$$

$\therefore\ 27|A| = 27I$ proved

Properties of Adjoint of Matrices.

1. $adj(A^T) = [adj(A)]^T$
2. $adj(kA) = K^{n-1}(adj\ A) \longrightarrow$ n-order of matrix : $K \in R$.
3. $adj(AB) = adj(A) \cdot adj(B)$.
4. $|adj\ A| = |A|^{(n-1)}$
5. $|adj[adj(A)]| = |A|^{(n-1)^2}$.
6. $adj(adj(A)) = |A|^{n-2} \cdot A$.

Q If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then verify ① $adj(A^T) = (adj\ A)^T$.
② $|adj(adj\ A)| = |A|$

Sol $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2.$

$adj\ A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

① $adj(A^T) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}.$

$= |4-6| = -2. = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ $\quad \therefore |adj\ A^T| = |A|$ Proved

② $adj(adj\ A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2.$ $\quad \therefore |adj(adj\ A)| = |A| = -2$ proved

$|A| = -2.$

Inverse of a Matrix :-

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

$|A| \neq 0$ — non singular.

Q Find the Inverse of $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Sol $\Delta = 1(0-0) + 1(0-0) + 1(0+1) = 1$. $\Delta \neq 0$ ✓

$$\text{Minor} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \qquad \text{Co-factor} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\operatorname{Adj}(A) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{Ans}$$

Properties of Inverse of Matrix.

1. $(A^{-1})^{-1} = A$.
2. $(AB)^{-1} = B^{-1}A^{-1} \rightsquigarrow (ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$.
4. $|A^{-1}| = |A|^{-1}$
5. $AA^{-1} = A^{-1}A = I$.
6. $(kA)^{-1} = \frac{1}{k} A^{-1}$

- Only square matrix have adjoint or Inverse.

Q For the matrix $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$ find x and y so that $A^2 + xI = yA$. Hence find A^{-1}.

Sol $A^2 = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$

$$\begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$16 + x = 3y$; $8 + 0 = y$; $56 + 0 = 7y$; $32 + x = 5y$.

$x = 8$. $\quad y = 8 \quad y = 8 \quad x = 40 - 32 = 8$

$A^2 + 8I = 8A \qquad \times A^{-1}$

$(A^{-1}A)A + 8A^{-1}I = 8A^{-1}A$

$IA + 8A^{-1} = 8I$

$8A^{-1} = (8I - A)$ $\therefore$ $A^{-1} = \frac{1}{8}\left[\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right] = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$ Ans

Q If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then prove that $A^2 - 4A - 5I = 0$ Find A^{-1}.

Sol $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

given $= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ proved

$A^2 - 4A - 5I = 0 \quad \times A^{-1}$

$(A^{-1}A)A - 4A^{-1}A - 5A^{-1}I = 0$

$A - 4I - 5A^{-1} = 0.$

$A^{-1} = \frac{1}{5}[A - 4I]$

$A^{-1} = \frac{1}{5}\left[\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right]$

$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$ Ans

Q If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ then show that $A^T \cdot A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

Sol $|A| = 1 + \tan^2 x \neq 0$

To find $A^{-1} \longrightarrow \text{adj}(A) = \begin{bmatrix} 1 & -\tan x \\ +\tan x & 1 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$A^{-1} = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

LHS $A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{1+\tan^2 x} & -\frac{\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} & \frac{1}{1+\tan^2 x} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{1+\tan^2 x} - \frac{\tan^2 x}{1+\tan^2 x} & -\frac{\tan x}{1+\tan^2 x} - \frac{\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} + \frac{\tan x}{1+\tan^2 x} & -\frac{\tan^2 x}{1+\tan^2 x} + \frac{1}{1+\tan^2 x} \end{bmatrix}$

$= \begin{bmatrix} \frac{1-\tan^2 x}{1+\tan^2 x} & -\frac{2\tan x}{1+\tan^2 x} \\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix}$

$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$ Verified.

Q 2016 Find the max value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

Sol $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix} = 1(\sin\theta\cos\theta - 0)$$
$$= \frac{1}{2}(2\sin\theta\cos\theta)$$
$$= \frac{\sin 2\theta}{2}$$

For max $\Rightarrow \sin 2\theta = 1$.

$\rightarrow \Delta = \frac{1}{2}$ Ans

Q 2014 Using property of determinants, prove

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

Q 2015 Find adjoint of $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ and show $A(\text{adj}A) = |A| I_3$.

Sol

$$\text{Adj}[A] = \begin{bmatrix} +\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix} = -1(1-4) + 2(2+4) - 2(-4-2)$$
$$= -1(-3) + 2(6) - 2(-6) = 3 + 12 + 12 = 27.$$

$$|A| I_3 = \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} \text{ LHS.}$$

RHS $\begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} = \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$ Proved

Q 2015 C
If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ then verify.

① $(AB)^{-1} = B^{-1}A^{-1}$

② $AA^{-1} = I$.

③ $|A^{-1}| = |A|^{-1}$

Sol $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$.

$\text{Adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & b \\ -c & a \end{bmatrix}$

$|A| = -8-3 = -11$

$|B| = 3-2 = 1$

$|AB| = 14-25 = -11$.

$\text{Adj}[A] = \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$

$\text{Adj}[B] = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$.

$\text{Adj}[AB] = \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}$.

① $[AB]^{-1} = \frac{1}{\Delta}\text{Adj}[AB]$

$= \frac{1}{-11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$ — RHS

$A^{-1} = \frac{1}{\Delta}\text{adj}[A] = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$.

$B^{-1} = \frac{1}{\Delta}\text{adj}[B] = \frac{1}{1}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix} \rightarrow$ LHS Proved

Q If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Show $A^2 = 4A - 3I$. Hence find A^{-1}.

Sol $A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$.

$4A - 3I = 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$.

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\therefore$ $\Delta = 4 - 1 = 3 \neq 0$ A^{-1} exist.

$A^2 = 4A - 3I \quad \times A^{-1}$

$A^{-1}A^2 = 4A^{-1}A - 3A^{-1}I$.

$(A^{-1}A)\cdot A = 4(A^{-1}A) - 3A^{-1}$

$IA = 4I - 3A^{-1}$ $\therefore$ $3A^{-1} = 4I - AI$.

$A^{-1} = \frac{4}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ Ans

Q 2015C In the interval $\frac{\pi}{2} < x < \pi$, find the value of x for which the matrix $\begin{bmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{bmatrix}$ is singular.

Sol $\Delta = 0$ Singular.

$4\sin^2 x - 3 = 0$. $\sin^2 x = \frac{3}{4}$. $\sin x = \frac{\sqrt{3}}{2}$. $\boxed{x = \frac{2\pi}{3}}$ Ans

Q 2013C If A is a square matrix of order 3 such that $|adj(A)| = 64$. Find $|A|$.

Sol $|adj(A)| = |A|^{n-1}$ $n = 3$.

$64 = |A|^{3-1} = |A|^2 = 8^2$. $\boxed{|A| = \pm 8}$ Ans

Q 2018 given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$ Compute A^{-1} and show that $2A^{-1} = 9I - A$.

Sol $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$|A| = 14 - 12 = 2 \neq 0$. A^{-1} exists.

clearly $adj(A) = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ $A^{-1} = \frac{1}{\Delta} Adj(A) = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$.

$9I - A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ $\therefore\ 2A^{-1}$ Proved

Delhi'2015

Q) If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$ Find $[A^T]^{-1}$

Sol $(A^T)^{-1} = (A^{-1})^T$.

2011

Q.
$$\left.\begin{aligned} x + 2y - 3z &= -4 \\ 2x + 3y + 2z &= 2 \\ 3x - 3y - 4z &= 11 \end{aligned}\right\}$$ solve the linear eq.

Sol $AX = B$.

$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$

$B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

To fin $A^{-1} \rightarrow \frac{1}{\Delta} \text{Adj } A$.

$-8-9 = -17$

$4+9$.

$$A^{-1} = \frac{1}{67}\begin{bmatrix} +\begin{vmatrix} 3 & 2 \\ -3 & -4 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ -3 & -4 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 3 & -4 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 3 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}$$

$\Delta = 1(-12+6) - 2(-8-6) - 3(-6-9)$

$\Delta = -6 + 28 + 45$

$\Delta = 67$.

$$A^{-1} = \frac{1}{67}\begin{bmatrix} -6 & +17 & 13 \\ 14 & 5 & 9 \\ -15 & 9 & -1 \end{bmatrix}$$

x, y, z $3, -2, 1$

$\begin{array}{r} 109 \\ -56 \end{array}$

$X = A^{-1}B$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & 9 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$x = \frac{24+34+143}{67} = \frac{201}{67} = 3.$

$y = \frac{-56+10+99}{67} = \frac{53}{67}.$

$x = 3$
$y = -2.$
$z = 1.$

APPLICATION of DETERMINANTS and MATRICES.

↳ Solving linear Equations. → Two Variable / → Three Variable

Equations, if solution exists – CONSISTENT.

Equations, if no solution exists – INCONSISTENT.

Let
$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$.
$a_3x + b_3y + c_3z = d_3$.

$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

$\boxed{AX = B.}$ $\times A^{-1}$

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B.$

$\boxed{X = A^{-1}B.}$

$\boxed{\begin{array}{c} AX = B. \\ \text{Find } |A| \end{array}}$

- $|A| \neq 0$
  System is Consistent.
  Solution exists.
  $X = A^{-1}B$
- $|A| = 0$
  then find $(\text{adj } A)B$.
  - $(\text{adj } A)B \neq 0$
    System is inconsistent and has no solution.
  - $(\text{adj } A)B = 0$
    - Consistent infinite No. of Solution.
    - Inconsistent has no Solution

Q Examine the Consistency of the system of equations. $x + 2y = 2$ and $2x + 3y = 3$.

Sol $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $|A| = 3 - 4 = -1 \neq 0$ System is Consistent $\therefore A^{-1}$ exists.

Q Repeat the above for $3x - y = 5$; $6x - 2y = 3$.

Sol $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$ Δ or $|A| = -6 + 6 = 0$ ⟶ A^{-1} does not exists.

$B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Now lets find $[\text{adj}(A)]B = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -10+3 \\ -30+9 \end{bmatrix} = \begin{bmatrix} -7 \\ -21 \end{bmatrix} \neq 0$

$\therefore$ System is inconsistent and has no Solution.

Q Solve the system of linear equations by matrix method. $4x - 3y = 3$: $3x - 5y = 7$.

Sol $4x - 3y = 3$
$3x - 5y = 7$

$A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

$|A| = \Delta = -20 + 9 = -11$

$X = A^{-1}B$

$A^{-1} = \frac{1}{\Delta} adj(A) = \frac{1}{-11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 5 & -3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 15-21 \\ 9-28 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -6 \\ -19 \end{bmatrix} = \begin{bmatrix} -6/11 \\ -19/11 \end{bmatrix}$

$x = -6/11$
$y = -19/11$

(Delhi 2016)

Q Using elementary transformations, find the Inverse of matrix $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and use it to solve the linear Eq

$8x + 4y + 3z = 19$.
$2x + y + z = 5$.
$x + 2y + 2z = 7$

Sol $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ Elementary Transformation. $A = IA$. $\longrightarrow I = A^{-1}A$.

$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A^{-1} \end{bmatrix} A$.

$R_3 \to R_3 - 2R_2$ $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} A$

$R_3 \to -\frac{1}{3} R_3$ $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & -1/3 \end{bmatrix} A$.

$R_1 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$.

$R_3 \to R_3 - 4R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 0 & 1 & 0 \\ 1 & -4 & 0 \end{bmatrix} A$.

$R_2 \to R_2 - 2R_1$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 0 & -1/3 & 2/3 \\ 1 & -4 & 0 \end{bmatrix} A$

$R_2 \to R_2 + R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -\frac{13}{3} & 2/3 \\ 1 & -4 & 0 \end{bmatrix} A$.

$R_3 \to -R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix} A$.

$I = A^{-1}A$.

$A^{-1} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix}$

(2) $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$.

$X = A^{-1}B$.

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix}\begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{10}{3} - \frac{7}{3} \\ 19 - \frac{65}{3} + \frac{34}{3} \\ -19 + 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$x = 1 \; : \; y = 2 \; : \; z = 1$. Ans