INDEFINITE INTEGRATION

It is the inverse process of differentiation :-

c is always there.

1. $\int x^n dx = \frac{x^{n+1}}{(n+1)}$; $n \neq -1$
2. $\int \frac{1}{x} dx = \log|x|$
3. $\int e^{ax} dx = \frac{e^{ax}}{a}$
4. $\int a^x dx = \frac{a^x}{\log a}$
5. $\int \sin ax \, dx = \frac{-\cos ax}{a}$
6. $\int \cos ax \, dx = \frac{\sin ax}{a}$
7. $\int \sec^2 ax \, dx = \frac{\tan x}{a}$
8. $\int \operatorname{cosec}^2 ax \, dx = -\frac{\cot x}{a}$
9. $\int \sec ax \tan ax \, dx = \frac{\sec ax}{a}$
10. $\int \operatorname{cosec} ax \cdot \cot ax \, dx = -\frac{\operatorname{cosec} ax}{a}$
11. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$
12. $\int \frac{1}{(1+x^2)} dx = \tan^{-1} x$
13. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$

Q $I = \int \sqrt[3]{x} \, dx$

$I = \int x^{1/3} dx = \int x^n dx$

$= \frac{x^{1+\frac{1}{3}}}{1+\frac{1}{3}} + c$

$= \frac{3x^{4/3}}{4} + c$ Ans

Q) $I = \int \frac{1}{x^{1/3}} dx$

$I = \int x^{-1/3} dx = \int x^n dx$

$= \frac{x^{-1/3+1}}{-\frac{1}{3}+1} = \frac{3x^{2/3}}{2} + c$ Ans

Q $I = \int 5^x dx$

Sol $= \int 5^x dx = \int a^x dx = \frac{a^x}{\log a}$

$= \frac{5^x}{\log 5} + c$ Ans

Theorem 1 $\frac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x)$.

Theorem 2. $\int K f(x)\,dx = K\int f(x)\,dx$.

Q $I = \int 3x^2\,dx$.

Sol $I = 3\int x^2\,dx$.

$= 3\cdot\frac{x^{2+1}}{2+1} + C$.

$= \frac{3}{3}x^3 + C \quad \therefore I = x^3 + C$

Ans

Q $I = \int 2^{(x+3)}\,dx$.

$= \int 2^x \cdot 2^3\,dx$.

$= 8\int 2^x\,dx$

$= 8\cdot\frac{2^x}{\log 2} + C$ Ans

Q $I = \int\left(3\sin x - 4\cos x + 5\sec^2 x - 2\,\text{cosec}^2 x\right)dx$.

Sol $I = 3\int \sin x\,dx - 4\int \cos x\,dx + 5\int \sec^2 x\,dx - 2\int \text{cosec}^2 x\,dx$.

$= -3\cos x - 4\sin x + 5\tan x + 2\cot x + C$. Ans

Q $I = \int\left(\frac{3x^4 - 5x^3 + 4x^2 - x + 2}{x^3}\right)dx$.

$= 3\int x\,dx - 5\int dx + 4\int \frac{1}{x}\,dx - \int x^{-2}\,dx + 2\int x^{-3}\,dx$.

$= \frac{3x^2}{2} - 5x + 4\log|x| - \frac{x^{-1}}{-1} + 2\,\frac{x^{-2}}{-2} + C$.

$= \frac{3}{2}x^2 - 5x + 4\log|x| + \frac{1}{x} - \frac{1}{x^2} + C$.

$\frac{x}{x^3} = \frac{1}{x^2} = \int x^{-2}\,dx$

$\frac{x^{-2+1}}{-2+1}$

$= \frac{x^{-1}}{-1} = -\frac{1}{x}$.

Q $I = \int \frac{(x^3 + 4x^2 - 3x - 2)}{(x+2)}\,dx$.

Sol Power of Numerator is greater than Denominator. So we may divide.

and If Pow of Denominator is " " Numerator thence do "partial fraction".

$$
\begin{array}{r|l}
x+2 & x^3 + 4x^2 - 3x - 2 \;(x^2 + 2x - 7 \\
 & \underline{-x^3 \mp 2x^2} \\
 & 2x^2 - 3x \\
 & \underline{-2x^2 \mp 4x} \\
 & -7x - 2 \\
 & \underline{-7x \mp 14} \\
 & 12
\end{array}
$$

$I = \int \frac{(x^3 + 4x^2 - 3x - 2)}{(x-2)}\,dx = \int\left(x^2 + 2x - 7 + \frac{12}{x+2}\right)dx$.

$$I = \int x^2 dx + 2\int x\,dx - 7\int dx + 12\int \frac{1}{(x+2)} dx$$

$$= \frac{x^3}{3} + \frac{2x^2}{2} - 7x + 12\log|x+2| + c$$

$$= \frac{x^3}{3} + x^2 - 7x + 12\log|x+2| + c$$

Q $I = \int \frac{(x^4+1)}{(x^2+1)}\,dx$

Sol $I = \frac{x^4+1}{x^2+1}$

$$x^2+1 \overline{\big)\, x^4+1\,}\big(x^2-1$$
$$\underline{-\cancel{x^4} + x^2}$$
$$-x^2+1$$
$$\underline{-\cancel{x^2} - 1}$$
$$2$$

$$I = \int \left(x^2 - 1 + \frac{2}{x^2+1}\right) dx$$

$$I = \int x^2 dx - \int dx + 2\int \frac{1}{x^2+1} dx$$

$$I = \frac{x^3}{3} - x + 2\tan^{-1}x + c$$

Q $I = \int \tan^2 x\, dx$

Sol $\tan^2 x = \sec^2 x - 1$

$$I = \int (\sec^2 x - 1)dx$$

$$= \int \sec^2 x\,dx - \int dx$$

$$I = \tan x - x + c$$

Q $I = \int \cot^2 x\, dx$

$\cot^2 x = \operatorname{cosec}^2 x - 1$

$$I = \int (\operatorname{cosec}^2 x - 1) dx$$

$$I = \int \operatorname{cosec}^2 x\,dx - \int dx$$

$$I = -\cot x - x + c$$

Q $I = \int \sin^2 \frac{x}{2}\, dx$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta$

$\cos 2\theta - 1 = -\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin^2\left(\frac{x}{2}\right) = \frac{1 - \cos 2(x/2)}{2}$

$\sin^2\left(\frac{x}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos x$

$$I = \int \left(\frac{1}{2} - \frac{1}{2}\cos x\right) dx$$

$$I = \frac{1}{2}[x - \sin x] + c$$

Q $I = \int \sqrt{1 - \sin 2x}\, dx$

Sol $1 - \sin 2x$

$1 - 2\sin x\cos x$

$\sin^2 x + \cos^2 x - 2\sin x\cos x$

$(\sin x - \cos x)^2$

$$I = \int \sqrt{(\sin x - \cos x)^2}\, dx$$

$$= \int (\sin x - \cos x)\, dx$$

$$I = -\cos x - \sin x + c$$

Q $I = \int \frac{\sin x}{1+\sin x}\, dx$

Sol $\frac{\sin x}{1+\sin x} = \frac{(\sin x + 1) - 1}{(1+\sin x)} = 1 - \frac{1}{(1+\sin x)}$

$$\frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)} = \frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} - \frac{\sin}{\cos x \cos x}$$

$$= \sec^2 x - \tan x \sec x$$

$$I = \int \frac{\sin x}{1+\sin x}\,dx = \int\left(1 - (\sec^2 x - \tan x \sec x)\right)dx$$

$$I = x - \tan x + \sec x + c$$

Q $I = \int \frac{\sec x}{(\sec x + \tan x)}\,dx$

Sol
$$= \int \frac{1/\cos x}{\frac{1}{\cos x} + \sin x/\cos x}\,dx$$

$$= \int \frac{1}{1+\sin x}\,dx \Rightarrow \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{1-\sin x}{\cos^2 x}\,dx$$

$$I = \int \sec^2 x\,dx - \int \sec x \tan x\,dx$$

$$I = \tan x - \sec x + c$$

Q $I = \int \frac{4 - 5\cos x}{\sin^2 x}\,dx$

Sol
$$I = \int\left(\frac{4}{\sin^2 x} - \frac{5\cos x}{\sin^2 x}\right)dx$$

$$I = \int 4\,\mathrm{cosec}^2 x\,dx - 5\int \cot x\,\mathrm{cosec}\,x\,dx$$

$$I = 4(-\cot x) - 5(-\mathrm{cosec}\,x) + c \quad : I = -4\cot x + 5\,\mathrm{cosec}\,x + c$$

Q $\int \left(\frac{1-\cos 2x}{1+\cos 2x}\right)dx$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$ — ①

$\cos 2\theta = \cos^2\theta - 1 + \cos^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$1 + \cos 2\theta = 2\cos^2\theta$ — ②

Sol Using Eq ① & ②

$$I = \int \frac{2\sin^2 x}{2\cos^2 x}\,dx$$

$$I = \int \tan^2 x\,dx \longrightarrow \tan^2\theta + 1 = \sec^2\theta$$

$$I = \int(\sec^2 x - 1)\,dx$$

$$I = \tan x - x + c$$

Q $I = \int \frac{1}{\sin^2 x \cos^2 x}\, dx$.

Sol $I = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) dx$.

$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx = +\tan x - \cot x + c$

Ans

Q $I = \int \left(\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\right) dx$.

$\cos 2\theta = \cos^2\theta - \sin^2\theta$.
$\cos 2\theta = \cos^2\theta - 1 + \cos^2\theta$
$\cos 2\theta = 2\cos^2\theta - 1$

Sol $(\cos 2x) - (\cos 2\alpha) = (2\cos^2 x - \cancel{1}) - (2\cos^2\alpha - \cancel{1})$

$= 2\cos^2 x - 2\cos^2\alpha$

$I = 2\int \frac{\cos^2 x - \cos^2 \alpha}{\cos x - \cos \alpha}\, dx = 2\int (\cos x + \cos\alpha)\, dx$.

$I = 2[\sin x + x\cos\alpha] + c$

Ans

INTEGRATION BY SUBSTITUTION:–

Rules :– ① $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)}$

$t = ax+b$

$\dfrac{dt}{dx} = a$

$dx = \dfrac{dt}{a}$

$\int t^n \left(\dfrac{dt}{a}\right) = \dfrac{1}{a}\int t^n \, dt$

$= \dfrac{1}{a}\left[\dfrac{t^{n+1}}{n+1}\right] = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$ Proved

② $\int \cos(ax+b)\, dx = -\dfrac{\sin(ax+b)}{a} + c$

$t = ax+b$

$dx = \dfrac{dt}{a}$

$\int \cos t \left(\dfrac{dt}{a}\right) = \dfrac{1}{a}\int \cos t = -\dfrac{1}{a}\sin t = -\dfrac{1}{a}\sin(ax+b) + c$ Ans

Q $I = \int (3x+5)^7 dx$.

Sol $\int (3x+5)^7 dx = \dfrac{(3x+5)^{(7+1)}}{3(7+1)} = \dfrac{(3x+5)^8}{24} + c$ Ans

Q $I = \int \sqrt{ax+b}\, dx$.

$I = \int (ax+b)^{1/2} dx = \dfrac{(ax+b)^{\left(\frac{1}{2}+1\right)}}{a\left(\frac{1}{2}+1\right)} = \dfrac{2(ax+b)^{3/2}}{3a} + c$ Ans

Q) $I = \int \sec^2(3x+5)\, dx$.

Sol $I = \int \sec^2(3x+5)\, dt$.

$= \dfrac{\tan(3x+5)}{3} + c$.

Q) $I = \int e^{(5x+3)} dx$.

Sol $I = \dfrac{e^{(5x+3)}}{5} + c$.

Q $I = \int \dfrac{\log x}{x} dx$.

Sol $I = \int \dfrac{\log x}{x} dx$.

$t = \log x$

$dt = \dfrac{1}{x} dx$

$I = \int t \cdot dt$.

$I = \dfrac{t^2}{2} + c = \dfrac{\log^2 x}{2} + c$ Ans

$I = \int \dfrac{\sec^2(\log x)}{x} \cdot dx$.

Sol $t = \log x \rightsquigarrow dt = \dfrac{1}{x} dx$.

$= \int \sec^2 t \, dt$.

$I = \tan t + c$

$I = \tan(\log x) + c$ Ans

Q $\int \frac{e^{\tan^{-1}x}}{(1+x^2)}dx$

Sol $t = \tan^{-1}x$.

$\frac{dt}{dx} = \frac{1}{1+x^2}$.

$dt = \frac{dx}{1+x^2}$.

$\int e^t dt = e^t + c$.

$I = e^{\tan^{-1}x} + c$ Ans

Q $\int \frac{\sin\sqrt{x}}{\sqrt{x}} \cdot dx$.

Sol $t = \sqrt{x}$. $\frac{dt}{dx} = \frac{1}{2\sqrt{x}}$ $\Rightarrow 2dt = \frac{dx}{\sqrt{x}}$.

$I = \int \sin t \cdot 2dt$.

$= -2\cos t + c$.

$= -2\cos\sqrt{x} + c$ Ans

Q $\int \cos^3 x \cdot \sin x \, dx$.

Sol $t = \cos x$.

$dt = \sin x dx$.

$= \int t^3 dt$.

$= \frac{t^4}{4} + c$.

$I = \frac{1}{4}\cos^4 x + c$ Ans

Q $\int \sqrt{\sin x} \cos x \, dx$.

$t = \sin x$.

$dt = \cos x \, dx$.

$\int \sqrt{t} \cdot dt = \int t^{1/2} dt$

$= \frac{t^{3/2}}{3/2} + c = \frac{2}{3} t^{3/2} + c$.

$I = \frac{2}{3}(\sin x)^{3/2} + c$ Ans

Q $\int \frac{\sin x}{(3+4\cos x)} dx$.

Sol $3 + 4\cos x = t$.

$0 + 4\sin x = \frac{dt}{dx}$.

$\sin x dx = \frac{dt}{4}$.

$\int \frac{1}{t}\frac{dt}{4} = \frac{1}{4}\log|t| + c$.

$= \frac{1}{4}\log|3+4\cos x| + c$.

Ans

$I = \int \frac{3x^2}{(1+x^6)} \cdot dx$.

Sol $t = x^3$

$dt = 3x^2 dx$.

$I = \int \frac{dt}{(1+t^2)}$.

$= \tan^{-1} t + c$

$I = \tan^{-1}(x^3) + c$ Ans

$I = \int \frac{x^8}{(1-x^3)^{1/3}} dx$.

Sol $t = 1 - x^3$.

$x^3 = 1 - t$.

$3x^2 = -\frac{dt}{dx}$

$x^2 dx = -\frac{1}{3}dt$.

$\int \frac{x^6 \cdot x^2 dx}{(1-x^3)^{1/3}} = \int \frac{(1-t)^2 \cdot (-\frac{1}{3}dt)}{t^{1/3}} = \frac{1}{3}\int \frac{(1+t^2-2t)dt}{t^{1/3}}$

$= -\frac{1}{3}\int (t^{-1/3} + t^{5/3} - 2t^{2/3})dt$

$= \left[\frac{t^{2/3}}{2} - \frac{1}{8}t^{8/3} + \frac{2}{5}t^{5/3}\right] + c$.

$$I = -\frac{1}{2}(1-x^3)^{2/3} - \frac{1}{8}(1-x^3)^{8/3} + \frac{2}{5}(1-x^3)^{5/3} + C$$

Q $I = \int \frac{(x-1)}{\sqrt{x-4}}\,dx.$

Sol $t = \sqrt{x-4}.$

$t^2 = x-4$

$x = t^2+4.$

$\frac{dx}{dt} = 2t.$

$I = \int \frac{(t^2+4-1)}{\cancel{t}} \cdot 2\cancel{t}\,dt.$

$I = 2\int (t^2+3)\,dt.$

$= 2\left[\frac{t^3}{3} + 3t\right]$

$= \frac{2}{3}t(t^2+9).$

$I = \frac{2}{3}\sqrt{x-4}\,(x-4+9) + c$

$= \frac{2}{3}\sqrt{(x-4)}\,(x+5) + c$

Q $\int \frac{(4x+3)}{\sqrt{2x^2+3x+1}}\,dx.$

Sol $t = \sqrt{2x^2+3x+1}$

$t^2 = 2x^2+3x+1$

$2t\,dt = (4x+3)dx$

$I = \int \frac{2\cancel{t}\,dt}{\cancel{t}}$

$I = 2t = 2\sqrt{2x^2+3x+1} + c.$

Q $\int \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) dx.$

Sol $\cos x + \sin x = t.$

$dt = (-\sin x + \cos x)\,dx.$

$\int \frac{dt}{t}$

$I = \log t + c.$

$I = \log|\cos x + \sin x| + c$

Q $\int \frac{\sec x}{\log(\sec x + \tan x)}\,dx.$

Sol $t = \log(\sec x + \tan x)$

$\frac{dt}{dx} = \frac{1}{(\sec x + \tan x)} \cdot (\sec x \tan x + \sec^2 x)$

$\frac{dt}{dx} = \frac{\sec x(\cancel{\tan x + \sec x})}{(\cancel{\sec x + \tan x})}$

$dt = \sec x\,dx$

$I = \int \frac{dt}{t} = \log t + c.$

$= \log|\log(\sec x + \tan x)| + c$

$I = \int \left(\frac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}\right) dx.$

Sol $t = a^2\sin^2 x + b^2\cos^2 x.$

$\frac{1}{a^2-b^2}\left[\log(a^2\sin^2 x + b^2\cos^2 x)\right] + c.$

$I = \int \frac{1}{(1+\tan x)}\,dx.$

Sol

$= \frac{1}{1+\frac{\sin x}{\cos x}} = \frac{2\cos x}{2[\cos x + \sin x]} = \frac{\cos x + \cos x + \sin x - \sin x}{2[\cos x + \sin x]} =$

$= \frac{\cos x + \sin x + \cos x - \sin x}{2(\cos x + \sin x)} = \frac{1}{2} + \frac{1}{2}\cdot\frac{\cos x - \sin x}{\cos x + \sin x}$

$$I = \int \left(\frac{1}{2} + \frac{1}{2} \frac{\cos x - \sin x}{\cos x + \sin x} \right) dx$$

$$I = \frac{1}{2} x + \frac{1}{2} \int \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right) dx$$

See above

$$I = \frac{x}{2} + \frac{1}{2} \log |\cos x + \sin x| dx + c$$

Q $\boxed{I = \int \frac{1}{1 + \cot x} \, dx}$

Sol $\frac{1}{1 + \cot x} = \frac{1}{1 + \frac{\cos x}{\sin x}} = \frac{\sin x}{(\sin x + \cos x)} \times \frac{2}{2} = \frac{(\sin x + \cos x) + (\sin x - \cos x)}{2(\sin x + \cos x)}$

See above

$$I = \int \frac{1}{2} dx - \int \left(\frac{\cos x - \sin x}{\sin x + \cos x} \right) dx$$

$$I = \frac{x}{2} - \frac{1}{2} \log |\sin x - \cos x| + c$$

Q $\boxed{I = \int \frac{\tan x}{\sec x + \cos x} dx}$

Sol $\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \cos x} = \int \frac{\sin x \, dx}{(1 + \cos^2 x)}$

$t = \cos x$

$dt = -\sin x \, dx$

$$= -\int \frac{dt}{1 + t^2} = -\tan^{-1} t + c$$

$$= -\tan^{-1} (\cos x) + c$$

Q $\boxed{I = \int \tan x \, dx}$

$$I = \int \frac{\sin x}{\cos x} dx$$

$t = \cos x$

$dt = -\sin x \, dx$

$$I = \int \frac{-1}{t} dt$$

$$I = -\log |t| + c$$

$$I = -\log |\cos x| + c$$

Q. $\boxed{I = \int \cot x \, dx}$

$$= \int \frac{\cos x}{\sin x} dx$$

$t = \sin x$

$dt = \cos x \, dx$

$$= \int \frac{1}{t} dt$$

$$I = \log |\sin x| + c$$

Q $\boxed{I = \int \sec x \, dx}$

$$I = \int \frac{\sec x [\sec x + \tan x]}{[\sec x + \tan x]} dx$$

$t = \sec x + \tan x$

$dt = (\sec^2 x + \sec x \tan x) dx$

$dt = \sec x (\sec x + \tan x) dx$

$$I = \int \frac{dt}{t}$$

$$I = \log |t| + c$$

$$I = \log |\sec x + \tan x| + c \quad \text{or} \quad I = \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + c$$

Q $\int \operatorname{Cosec} x \, dx$.

Sol $I = \int \frac{\operatorname{Cosec} x\,(\operatorname{Cosec} x - \operatorname{Cot} x)}{(\operatorname{Cosec} x - \operatorname{Cot} x)}\, dx$.

$t = \operatorname{Cosec} x - \operatorname{Cot} x$.

$dt = (\operatorname{Cosec}^2 x - \operatorname{Cosec} x \operatorname{Cot} x)\, dx$.

$I = \int \frac{dt}{t} = \log|t| + c$.

$I = \log|\operatorname{Cosec} x - \operatorname{Cot} x| + c$ Ans or $I = \log\left|\tan\frac{x}{2}\right| + c$.

NOTE

$\int \operatorname{Sin} x \, dx = -\operatorname{Cos} x$

$\int \operatorname{Cos} x \, dx = \operatorname{Sin} x$.

$\int \tan x \, dx = -\log|\operatorname{Cos} x|$.

$\int \operatorname{Cot} x \, dx = \log|\operatorname{Sin} x|$

$\int \operatorname{Sec} x \, dx = \log|\operatorname{Sec} x + \tan x| = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|$

$\rightarrow \int \operatorname{Cosec} x \, dx = \log|\operatorname{Cosec} x - \operatorname{Cot} x| = \log\left|\tan\frac{x}{2}\right|$

Q $I = \int \frac{(1 + \operatorname{Cos} x)}{(1 - \operatorname{Cos} x)}\, dx$

$\operatorname{Cos} 2\theta = \operatorname{Cos}^2\theta - \operatorname{Sin}^2\theta$.
$\operatorname{Cos} 2\theta = \operatorname{Cos}^2\theta - 1 + \operatorname{Cos}^2\theta$.
$\operatorname{Cos} 2\theta = 2\operatorname{Cos}^2\theta - 1$
$1 + \operatorname{Cos} 2\theta = 2\operatorname{Cos}^2\theta$.
$1 + \operatorname{Cos} x = 2\operatorname{Cos}^2\left(\frac{x}{2}\right)$
$1 - \operatorname{Cos} x = 2\operatorname{Sin}^2(x/2)$.

Sol $I = \int \frac{2\operatorname{Cos}^2(x/2)}{2\operatorname{Sin}^2(x/2)}\, dx$.

$= \int \operatorname{Cot}^2\left(\frac{x}{2}\right) = \int \left(\operatorname{Cosec}^2\frac{x}{2} - 1\right) dx$.

$= \int \operatorname{Cosec}^2\left(\frac{x}{2}\right) dx - \int dx$.

$t = \frac{x}{2}$. $=$ $dt = \frac{1}{2}\, dx$.

$= 2\int \operatorname{Cosec}^2 t \cdot dt - \int dx$.

$= -2\operatorname{Cot} t - x + c$.

$I. = -2\operatorname{Cot}\left(\frac{x}{2}\right) - x + c$. Ans.

$I = \int \frac{x + \sqrt{x+1}}{(x+2)}\, dx$.

Sol $t = \sqrt{x+1}$

$t^2 = x + 1 \rightarrow x = t^2 - 1$

$2t\,dt = dx$.

$I = 2\int \left(\frac{t^2 - 1 + t}{t^2 - 1 + 2}\right) t\,dt. = 2\int \frac{(t^2 + t - 1)t}{(t^2 + 1)} \cdot dt$.

$I = 2\int \frac{t^3 + t^2 - t}{t^2 + 1}\, dt$.

$$t^2+1 \overline{) t^3+t^2-t} (t+1$$
$$\underline{-t^3 \pm t}$$
$$t^2 - 2t$$
$$\underline{-t^2 \pm 1}$$
$$-2t-1$$

$$I = 2\int (t+1) - \frac{(2t+1)}{(t^2+1)}\, dt.$$

$$I = 2\int \left[t+1 - 2\left(\frac{t}{t^2+1}\right) - \frac{1}{(t^2+1)} \right] dt.$$

$$I = 2\int t\,dt + 2\int dt - 4\int \frac{t}{t^2+1}\,dt - 2\int \frac{1}{t^2+1}\,dt.$$

NOTE $\int \frac{t}{t^2+1}\,dx$.

$t^2+1 = a$.

$\frac{da}{dt} = 2t$.

$\frac{da}{2} = t\,dt$.

$\frac{1}{2}\int \frac{da}{a}$.

$\frac{1}{2}\log a$

$= \frac{1}{2}\log|t^2+1|$

$$I = 2\frac{t^2}{2} + 2t - 4\cdot\frac{1}{2}\log|t^2+1| - 2\tan^{-1}\sqrt{t+1} + c.$$

$$I = (x+1) + 2\sqrt{x+1} - 2\log|x+2| - 2\tan^{-1}\sqrt{\sqrt{x+1}+1} + c$$

Ans

NOTE
1. $2\sin^2\frac{x}{2} = 1-\cos x$.
2. $2\cdot\cos^2\frac{x}{2} = 1-\sin x$.
3. $2\sin a\cos b = \sin(a+b)+\sin(a-b)$
4. $2\cos a\sin b = \sin(a+b) - \sin(a-b)$
5. $2\cos a\cos b = \cos(a+b) + \cos(a-b)$
6. $2\sin a\sin b = \cos(a-b) - \cos(a+b)$.
7. $\tan^2 x = \sec^2 x - 1$.

Q $I = \int \sin^2\frac{x}{2}\,dx$.

Sol
$$I = \frac{1}{2}\int (1-\cos x)\,dx.$$
$$= \frac{1}{2}\left[x - \sin x\right] + C.$$

Q $I = \int \tan^2\frac{x}{2}\,dx$.

$$I = \int \left(\sec^2\left(\frac{x}{2}\right) - 1\right) dx.$$
$$= \frac{\tan\left(\frac{x}{2}\right)}{1/2} - x + c.$$
$$I = 2\tan\left(\frac{x}{2}\right) - x + c$$ Ans

$I = \int \sin^7 x\,dx$

Sol
$$= \int \sin^6 x \cdot \sin x\,dx.$$
$$= \int [\sin^2 x]^3 \cdot \sin x\,dx.$$
$$= \int [1-\cos^2 x]^3 \sin x\,dx.$$
$\cos x = t$.
$dt = -\sin x\,dx$.

$I = \int \sin^3(2x+1)$.

Sol
$$I = \int \sin^2(2x+1)\cdot\sin(2x+1)\,dx.$$
$$= \int [1-\cos^2(2x+1)]\cdot\sin(2x+1)\,dx.$$
$t = \cos(2x+1)$.
$dt = -2\sin(2x+1)\,dx$

$$I = -\int (1-t^2)^3 dt.$$

$$= -\int (1 - t^6 - 3t^2 + 3t^4) dt.$$

$$= -t + \frac{t^7}{7} + \frac{3t^3}{3} - \frac{3t^5}{5} + C.$$

$$I = -\cos x + \frac{\cos^7 x}{7} + \cos^3 x - \frac{3}{5}\cos^5 x + C.$$

Ans

$$= -\frac{1}{2}\int (1-t^2)\cdot dt$$

$$= -\frac{1}{2}\left[t - \frac{t^3}{3}\right] + c$$

$$= -\frac{1}{2}\left[\cos(2x+1) - \frac{\cos^3(2x+1)}{3}\right] + c$$

$$I = -\frac{1}{2}\cos(2x+1) + \frac{1}{6}\cos^3(2x+1) + c.$$

Ans

Q Evaluate $\int \cos mx \cdot \cos nx$ where ① $m \neq n$ ② $m = n$.

Sol $m \neq n$

$$\int \cos mx \cdot \cos nx \, dx = \frac{1}{2}\int (\cos(m+n)x + \cos(m-n)x) \cdot dx.$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{(m+n)} + \frac{\sin(m-n)x}{(m-n)}\right] + C$$

Ans

$m = n$

$$\int \cos mx \cdot \cos mx \, dx. = \int \cos^2 mx \, dx.$$

$$= \frac{1}{2}\int (1 + \cos 2mx) \, dx.$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2mx \cdot dx.$$

$$I = \frac{x}{2} + \frac{\sin 2mx}{4m} + c.$$

Q $I = \int \cos x \cos 2x \cos 3x \cdot dx.$

Sol

$$I = \frac{1}{2}\int (2\cos x \cos 2x) \cos 3x \cdot dx.$$

$$= \frac{1}{2}\int (\cos 3x + \cos x) \cos 3x \cdot dx.$$

$$= \frac{1}{2}\int (\cos^2 3x \, dx + \cos 3x \cos x) dx.$$

$$= \frac{1}{2}\int \cos^2 3x \, dx + \frac{\cos 4x + \cos 2x}{2} \cdot dx.$$

$$= \frac{1}{2}\int \frac{(1 - \cos 6x)}{2} \cdot dx + \frac{1}{4}\int (\cos 4x + \cos 2x) dx$$

$$= \frac{1}{4}\left[x - \frac{\sin 6x}{6}\right] + \frac{1}{4}\left[\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right] + c.$$

$$= \frac{x}{4} - \frac{\sin 6x}{24} + \frac{\sin 4x}{16} + \frac{\sin 2x}{8} + c$$

Ans

Q $I = \int \frac{\sin x}{\sin(x-\alpha)}\,dx$

Sol $x-\alpha = t$

$x = t+\alpha$

$dx = dt$

$$I = \int \frac{\sin(t+\alpha)}{\sin t}\,dt$$

$$= \int \frac{\sin t\cos\alpha + \sin\alpha\cos t}{\sin t}\,dt$$

$$= \int (\cos\alpha + \sin\alpha\cot t)\,dt$$

$$= \cos\alpha\int dt + \sin\alpha\int \cot t\,dt$$

$$= t\cos\alpha + \sin\alpha\log|\sin t| + C$$

$$= (x-\alpha)\cos\alpha + \sin\alpha\cdot\log|\sin(x-\alpha)| + C$$

Q $I = \int\sqrt{1+\sin x}\,dx$

Sol $1+\sin x$

$$\underbrace{\left[\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)\right]}_{1} + \underbrace{\left[2\sin\frac{x}{2}\cos\frac{x}{2}\right]}_{\sin x}$$

$$\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2$$

$$I = \int\sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}\,dx$$

$$= \int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)dx$$

$$I = -2\cos x/2 + 2\sin x/2 + C$$

Q $I = \int\frac{dx}{\sin x+\cos x}$

Sol $\frac{1/\sqrt{2}}{1/\sqrt{2}} \times \frac{1}{\sin x + \cos x}$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sin\left(x+\frac{\pi}{4}\right)}\right]$$

$$I = \frac{1}{\sqrt{2}}\int\operatorname{cosec}\left(x+\frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\frac{(x+\pi/4)}{2}\right| + C$$

$$I = \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right| + C$$

Q $I = \int\frac{1}{\sqrt{1-\sin x}}\,dx$

$1-\sin x$

$$\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)$$

$$= \left(\sin\frac{x}{2} - \cos x/2\right)^2$$

$$I = \int\frac{1}{\sin\left(\frac{x}{2}\right)-\cos\left(\frac{x}{2}\right)}\,dx$$

$$I = \frac{1}{\sqrt{2}} \int \frac{1}{\frac{1}{\sqrt{2}} \sin\left(\frac{x}{2}\right) - \frac{1}{\sqrt{2}} \cos\frac{x}{2}} \cdot dx .$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\cos\left(\frac{\pi}{4}\right) \sin\frac{x}{2} - \sin\left(\frac{\pi}{4}\right) \cos\frac{x}{2}} \cdot dx .$$

$$\frac{1}{\sqrt{2}} \int \frac{1}{\sin\left(\frac{x}{2} - \frac{\pi}{4}\right)} dx . = \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(\frac{x}{2} - \frac{\pi}{4}\right) dx .$$

$$= \frac{1}{\sqrt{2}} \frac{\log\left| \tan \frac{\left(\frac{x}{2} - \frac{\pi}{4}\right)}{2} \right|}{1/2} + C .$$

$$= \frac{2}{\sqrt{2}} \log\left| \tan\left(\frac{x}{4} - \frac{\pi}{8}\right) \right| + C . \quad \underline{\underline{\text{Ans}}}$$

INTEGRATION BY PARTS :-

Parts of Type of Equation :-
1. INVERSE. I
2. LOG L
3. ALGEBRAIC. A
4. TRIGO. T
5. EXP. E

ILATE

• If u and v are two functions of 'x'.

$$\boxed{\int u\cdot v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx}$$

ILATE

Q $I = \int x^2 \sin x\,dx$

Sol $x^2 \to$ Alg

$\sin x \to$ Tri.

$I = \int u\cdot v\,dx$

$u = x^2 \;:\; v = \sin x$

$\int u\cdot v\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$

$\frac{du}{dx} = 2x \;:\; \int v\,dx = \int \sin x\,dx = -\cos x$

$\int x^2 \sin x\,dx = x^2(-\cos x) - \int 2x\cdot(-\cos x)\cdot dx$

$= -x^2\cos x + 2\underbrace{\int x\cos x\,dx}$

$u = x$

$v = \cos x$

$\int u\cdot v\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$

$= x\int \cos x\,dx - \int\left(\frac{d}{dx}(x)\int \cos x\,dx\right)dx$

$= x\sin x - \int 1\cdot \sin x\cdot dx$

$= x\sin x + \cos x$

$\int x^2\sin x\,dx = -x^2\cos x + 2[x\sin x + \cos x] + c$

$= -x^2\cos x + 2x\sin x + 2\cos x + c$

Q) $I = \int x^n \log x\,dx$

Sol $v = x^n \;:\; u = \log x$

$\int v\,dx = \int x^n\,dx = \frac{x^{n+1}}{n+1}$

$\frac{du}{dx} = \frac{1}{x}$

$\int u\cdot v\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$

$= \log x \cdot \frac{x^{n+1}}{n+1} - \int \frac{1}{x}\cdot\frac{x^{n+1}}{n+1}\,dx$

$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^{n+\cancel{1}-\cancel{1}}\,dx$

$= \frac{1}{n+1}\left[x^{n+1}\log x - \frac{x^{n+1}}{n+1}\right] + c$

$I = \frac{1}{(n+1)^2}\left[(n+1)x^{(n+1)}\log x - x^{(n+1)}\right] + c$

Q $I = \int x \sin 2x \cdot dx$

ANS → $-\frac{x \cos 2x}{2} + \frac{1}{4}\sin 2x + c$

Sol

Q $I = \int x \cos^2 x \cdot dx$

Sol $\cos^2 x = \left(\frac{1 + \cos 2x}{2}\right)$

I LATE

$$I = \frac{1}{2}\int x(1+\cos 2x)\,dx$$

$$I = \frac{1}{2}\left[\int x\,dx + \int x \cos 2x \cdot dx\right]$$

$$I = \frac{1}{2}\left[\frac{x^2}{2} + \left[x\int \cos 2x \cdot dx - \int\left(\frac{d(x)}{dx}\int (\cos 2x)\,dx\right)\right]\right]$$

$$I = \frac{1}{2}\left[\frac{x^2}{2} + x\frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2}\,dx\right]$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + c \quad \text{Ans}$$

Q $I = \int \log x \cdot dx$

Sol
$$I = \int \log x \cdot 1\,dx$$

$$= \log x \int 1\,dx - \int\left[\frac{d}{dx}(\log x)\int 1 \cdot dx\right]dx$$

$$= x \log x - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \cdot dx$$

$$I = x \log x - x + c \quad \text{Ans}$$

Q $I = \int \log(1+x^2)\,dx$

Sol
$$I = \int \log(1+x^2) \cdot 1\,dx$$

$$= \log(1+x^2)\int 1\,dx - \int\left[\frac{d}{dx}\left(\log(1+x^2)\right)\int 1 \cdot dx\right]dx$$

$$= \log(1+x^2) \cdot x - \int \frac{1}{(1+x^2)} \cdot 2x \cdot x\;dx$$

$$= x\log(1+x^2) - 2\int \frac{x^2}{1+x^2} \cdot dx$$

$$\frac{(x^2+1)-1}{(1+x^2)} = 1 - \frac{1}{(1+x^2)}$$

$$= x\log(1+x^2) - 2\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$x\log(1+x^2) - 2\left[x - \tan^{-1} x\right] + c \quad \text{Ans}$$

$$I = \int (\log x)^2 \cdot dx$$

Sol $I = \int (\log x)^2 \cdot 1\, dx = (\log x)^2 \int 1\, dx - \int \left(\frac{d}{dx}(\log x)^2 \int 1 \cdot dx\right) dx$

$= x(\log x)^2 - \int \frac{2\log x}{\cancel{x}} \cdot \cancel{x} \cdot dx$

$= x(\log x)^2 - 2\int \log x \cdot dx$

$= x(\log x)^2 - 2[x\log x - x] + c$

Q $I = \int \frac{\log x}{x^2} \cdot dx$ ILATE

Sol $\int \frac{\log x}{x^2} \cdot dx = \int \log x \cdot \frac{1}{x^2} \cdot dx$

$= \log x \int x^{-2} dx - \int \left(\frac{d}{dx}(\log x) \int x^{-2} dx\right) dx$

$= \log x \frac{x^{-1}}{-1} - \int \frac{1}{x} \cdot \frac{x^{-1}}{(-1)} \cdot dx \Rightarrow -\frac{\log x}{x} + \int x^{-2} dx$

$= -\frac{\log x}{x} - \frac{1}{x} + c$

Q $I = \int e^{2x} \sin x \cdot dx$

Sol $I = \int \underbrace{e^{2x}}_{② = v} \cdot \underbrace{\sin x}_{① = u} dx$

ILATE $u = \sin x$ $\quad v = e^{2x}$

$\frac{du}{dx} = \cos x$ $\quad \int v\,dx = \frac{e^{2x}}{2}$

$I = \sin x \cdot \frac{e^{2x}}{2} + \int \cos x \cdot \frac{e^{2x}}{2} dx$

$I = \int e^{2x} \sin x = \frac{1}{2} e^{2x} \sin x + \frac{1}{2} \int \cos x \cdot e^{2x} dx$

$\int e^{2x} \sin x\, dx = \frac{1}{2} e^{2x} \sin x + \frac{1}{2}\left[\cos x \frac{e^{2x}}{2} - \int \sin x \cdot \frac{e^{2x}}{2} dx\right]$

$\underbrace{\int e^{2x} \sin x\, dx}= \frac{1}{2} e^{2x} \sin x + \frac{e^{2x}}{4} \cos x - \frac{1}{4}\left[\underbrace{\int e^{2x} \sin x\, dx}\right]$

$\left(1 + \frac{1}{4}\right) \int e^{2x} \sin x\, dx = \frac{1}{2} e^{2x} \sin x - \frac{e^{2x}}{4} \cos x$

$I = \int e^{2x} \sin x\, dx = 2 e^{2x} \sin x - e^{2x} \cos x$

$I = \frac{1}{5} e^{2x} [2 \sin x - \cos x] + c$

Q $\int \left(\frac{x-\sin x}{1-\cos x}\right) dx$.

Sol $\int \frac{x-\sin x}{2\sin^2(\frac{x}{2})} dx$

$$= \int \frac{x}{2\sin^2(\frac{x}{2})} dx - \int \frac{\sin x}{2\sin^2(\frac{x}{2})} . dx .$$

$$\frac{\sin 2(\frac{x}{2}) = \cancel{2}\sin\frac{x}{2}\cos\frac{x}{2}}{\cancel{2}\sin^{\cancel{2}}(x/2)}$$

$$= \frac{1}{2}\int x \operatorname{Cosec}^2 \frac{x}{2} dx - \int \cot(\frac{x}{2}) dx .$$

ILATE

$$= \frac{1}{2}\left[x\int \operatorname{Cosec}^2(\frac{x}{2}) - \int\left[\frac{d}{dx}x\int(\operatorname{Cosec}^2\frac{x}{2})dx\right]dx\right] - \int \cot(\frac{x}{2}) dx$$

$$= \frac{1}{2}\left[-x\cot(\frac{x}{2})\times\frac{1}{(1/2)} + \int 1\,\frac{\cot(x/2)}{1/2} dx\right] - \int \cot(\frac{x}{2}) dx .$$

$$= -x\cot(\frac{x}{2}) + \cancel{\int \cot(x/2)dx} - \cancel{\int \cot(x/2)dx} = -x\cot(\frac{x}{2}) + c .$$

Q $I = \int x^2 \sin^{-1}x\, dx$.

Sol $= \sin^{-1}x\int x^2 dx - \int\left(\frac{d}{dx}(\sin^{-1}x)\int x^2 dx\right)dx .$

$$= \sin^{-1}x \cdot \frac{x^3}{3} - \int\left(\frac{1}{\sqrt{1-x^2}} \cdot \frac{x^3}{3}\right) dx .$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}} . dx .$$

$t = \sqrt{1-x^2}$

$t^2 = 1-x^2$.

$\cancel{2}t\frac{dt}{dx} = -\cancel{2}x$.

$t\,dt = x\,dx$

$$= \quad " \quad -\frac{1}{3}\int -\frac{x^2}{\cancel{t}}\cancel{t}\,dt .$$

$$" \quad -\frac{1}{3}\int (1-t^2) dt .$$

$$" \quad -\frac{1}{3}\left[t - \frac{t^3}{3}\right]$$

$$" \quad -\frac{1}{3}\left[\sqrt{1-x^3} - \frac{1}{3}(1-x^3)^{1/3}\right]$$

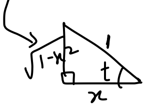
$$= \frac{x^3}{3}\sin^{-1}x . - \frac{\sqrt{1-x^3}}{3} - \frac{1}{9}(1-x^3)^{1/3} . + C$$

Q $I = \int \cos^{-1}x\, dx$.

Sol $\cos^{-1}x = t$.

$x = \cos t$.

$dx = -\sin t\, dt$



$$I = -\int t \cdot \sin t\, dt$$

$$I = -\int t \sin t\, dt .$$

$$I = -\left[t(-\cos t) + \int[1 \cdot \cos t\, dt]\right]$$

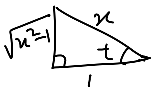
$$I = t\cos t - \sin t.$$
$$I = x\cos^{-1}x - \frac{\sqrt{1-x^2}}{1} + C.$$

Q $\int \sec^{-1}x \, dx.$

Sol $\sec^{-1}x = t.$

$x = \sec t$

$dx = \sec t \tan t . dt.$



$\cos t = \frac{1}{x}.$

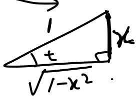
$$I = \int t(\sec t \tan t)\,dt.$$
$$= t\int \sec t \tan t \, dx - \int\left(\frac{d}{dt}(t)\int \sec t \tan t\,dt\right)dt.$$
$$= t\sec t - \int \sec t\, dt.$$
$$= t\sec t - \log|\sec t + \tan t| + C.$$
$$= x\sin^{-1}x - \log\left|x + \frac{\sqrt{x^2-1}}{1}\right| + C$$

Q $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}} . dx.$

Sol $\sin^{-1}x = t.$

$\sin t = \frac{(x) \to P}{1 \to H}$

$\frac{dx}{dt} = \cos t.$

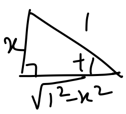


$$I = \int \frac{t\cos t}{(1-\sin^2 t)^{3/2}}\,dt. = \int \frac{t\cancel{\cos t}}{\cos^{\cancel{3}2} t}.dt.$$
$$I = \int \frac{t}{\cos^2 t}.dt =$$
$$I = \int t\sec^2 t . dt.$$
$$I = t.\tan t - \int 1.\tan t\,dt.$$
$$= t\tan t + \log(\cos t).dt.$$
$$= \frac{x}{\sqrt{1-x^2}}.\sin^{-1}x + \log\left|\frac{\sqrt{1-x^2}}{1}\right| + c$$

Q $I = \int \sin^{-1}(3x-4x^3)\,dx.$

Sol $x = \sin t.$

$dx = \cos t . dt.$



$$I = \int \sin^{-1}(3x-4x^3)\,dx.$$
$$= \int \sin^{-1}(3\sin t - 4\sin^3 t).\cos t\,dt.$$
$$= \int \sin^{-1}(\sin 3t)\cos t\,dt.$$
$$= \int 3t\cos t\,dt. = 3\left[t\sin t - \int 1.\sin t.dt\right]$$
$$= 3\left[t\sin t + \cos t\right] + c.$$
$$= 3\left[x\sin^{-1}x + \frac{\sqrt{1-x^2}}{1}\right] + c.$$
$$= 3x\sin^{-1}x + 3\sqrt{1-x^2} + c$$